Furry Ball Therom what it has to do with cats and 3D space.

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1 Introduction

Consider a theoretical function,

$$\langle x\prime, y\prime, z\prime\rangle = f(x, y, z)$$

such that this result is always orthogonal to the input, and that a minute adjustment in the direction of input will result in a similarly proportioned adjustment in the output. There are many ways to analyse this problem, however ultimately, this function can not exist.

2 4D Cross Product

Initially, it seemed that the 4D cross product may provide a useful result in solving this problem. When looking at the problem in 2D, a function for finding orthogonal exists and can be written as:

$$f(x, y) = \times_3(\langle x, y, 0\rangle, \langle 0, 0, 1\rangle)$$

This satisfies the above requirements, but only in 2D. To achieve the same in 3D, one might write:

$$f(x, y, z) = \times_4(\langle x, y, z, 0\rangle, \langle 0, 0, 0, 1\rangle, \langle ?, ?, ?, ?\rangle)$$

However, the extension of the cross product to 4D requires an additional vector to constrain the result. There are some basic requirements for this unknown; it cannot be the same as either the first or second input. Thus at least one of its 3D terms must always be non-zero, and its 4th term must also always be non-zero.

$$f(x, y, z) = \times_4(\langle x, y, z, 0\rangle, \langle 0, 0, 0, 1\rangle, \langle 1, 1, 1, 1\rangle)$$

This function provides some very positive results. Yet, there is a major problem when x = y = z. We can solve the 4D cross product[1] to analyse this result:

$$\times_4(U,V,W) = i\begin{vmatrix} U_1 & U_2 & U_3 \\ V_1 & V_2 & V_3 \\ W_1 & W_2 & W_3 \end{vmatrix} - j\begin{vmatrix} U_0 & U_2 & U_3 \\ V_0 & V_2 & V_3 \\ W_0 & W_2 & W_3 \end{vmatrix} + k\begin{vmatrix} U_0 & U_1 & U_3 \\ V_0 & V_1 & V_3 \\ W_0 & W_1 & W_3 \end{vmatrix} - l\begin{vmatrix} U_0 & U_1 & U_2 \\ V_0 & V_1 & V_2 \\ W_0 & W_1 & W_2 \end{vmatrix}$$

Solving this function in terms of the input tangent, we will get a function that looks like this:

$$f(x,y,z) = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

It is clear from this skew symmetric matrix, that when x = y = z, the result will be zero. An example tube rendered with normals using this function can be seen in Figure 1. This tube can be manipulated to show this problem.

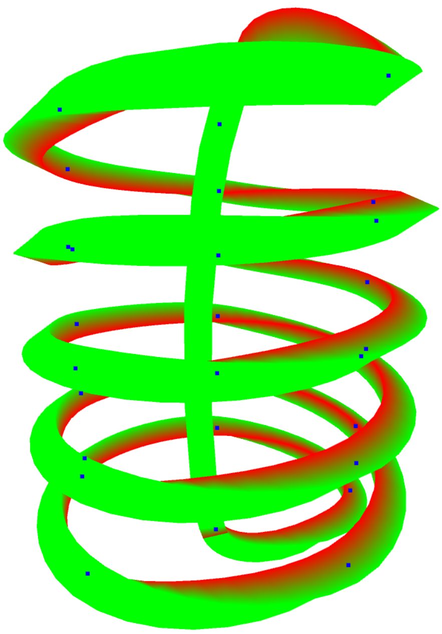

Figure 1: A triangle tube around a 'spring' shaped spline with normals calculated using the 4D cross product. The red edge marks the normals generated.

3 The problem with cats

Consider a cat with fur oriented in 3-space (a cat in 4-space incidentally, does not have this problem). We take any point on the cat's surface. At this point, we consider the surface normal vector - and typically - there will also exist a cat hair, orthogonal to this surface normal. We may even begin to think, that given any surface normal on a cat, there is a corresponding orthogonal hair...

However, this is not the case! There are two points on a cat where no orthogonal hairs exist - at the tip of the nose, and at the end of the tail. At the nose of the cat, all hairs point down the cats body. At the end of the tail, all the hairs point away from the cat - ie, in the same direction as the tip of the tail surface normal.

This problem is referred to as the Hairy Ball Theroem[2]. Given a 3D sphere, a smoothly varying vector field over the surface of the sphere has to have at least one point (but typically two points) where the field is 0, as seen in Figure 2. The function we have defined above clearly has this problem.

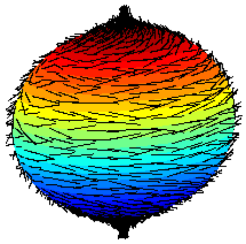

Figure 2: A failed attempt to comb a hairy ball flat, leaving two tufts at the top and bottom[2].

4 Further implications

Interestingly enough, this has far greater implications than one might initially expect.

> "Another consequence of this theorem applies to wind blowing over the surface of the earth. Either there have to be places where there is no wind or at some places the wind would be blowing in opposite directions, directly adjacent to each other. This corresponds nicely to what we know about hurricanes and tornados, that in there centre the eye there is no wind. Even if we were to imagine one big flow of wind going over the earths surface in a direction along the equator, then still there would be areas without wind at the poles." [3]

Any shape which can have its surface deformed to a sphere will exhibit this problem. However, a doughnut shape (see figure 3) does not have this problem, and it is possible to map every surface normal on a doughnut to a corresponding orthogonal vector.

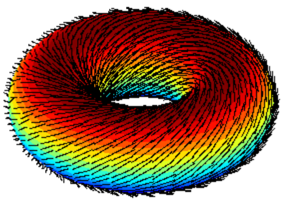

Figure 3: A hairy doughnut, on the other hand, is quite easily combable[2].

5 Finally

The results of the 4D cross product could still potentially simplify a given solution to this problem. However, the result will always be on the plane which is orthogonal to the input tangent, thus the problem can be represented entirely in 3-space.

Further investigation is required to provide a working example.

References

[1] Steven Richard Hollasch. *Four-Space Visualization of 4D Objects.* PhD thesis, Arizona State University, 1991. Available from: `http://steve.hollasch.net/thesis/index.html`.

[2] Wikipedia. Hairy ball theorem — wikipedia, the free encyclopedia, 2007. [Online; accessed April-2007]. Available from: `http://en.wikipedia.org/w/index.php?title=Hairy_ball_theorem&oldid=118499256`.

[3] Matthijs Sypkens Smit. Hairy ball theorem, 2007. [Online; accessed April-2007]. Available from: `http://matthijs.mired.nl/blog/?p=18`.